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One-loop unitarity of string theories in a constant external background and their Seiberg-Witten limit

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Abstract: Perturbative spectra and related factorization properties of one-loop open string amplitudes in the presence of a constant external background B are analysed in detail. While the pattern of the closed string spectrum, obtained after a careful study of the properly symmetrized amplitudes, turns out to be unaffected by the presence of B, a series of double open-string poles, which would be absent when B is turned off, can couple owing to a partial symmetry loss. These features are studied first in a bosonic setting and then generalized in the more satisfactory superstring context. When the background is of an "electric" type, a classical perturbative instability is produced beyond a critical value of the electric field. In the Seiberg-Witten limit this instability is the origin of the unphysical tachyonic cut occurring in the non-planar amplitudes of the corresponding noncommutative field theories.

Keywords: Bosonic Strings, Superstrings and Heterotic Strings, D-branes.

Contents

1. Introduction

String theories in the presence of a constant background field (B-field) have been often considered in the recent literature; in particular Seiberg and Witten [1] have shown they give rise in a suitable limit to field theories defined on a space of noncommuting variables (NCFT).

In turn NCFT may suffer from the lack of covariance and, when the B-field is of electric type, their spectrum has severe difficulties with respect to perturbative S-matrix unitarity [2, 3, 4]. Usual analytic properties of amplitudes are lost as well, as a consequence of non-locality.

Many of these unpleasant features are not shared by the parent string theory [5, 6]; therefore it looks interesting to re-examine the spectrum of the latter as a first step in order to point out where possible differences are generated. Such an analysis does not seem to have been carried out in full detail, at least to our knowledge, in spite of the huge amount of literature on the subject (see for instance [7]–[15]).

A first approach to this problem was undertaken in ref. [16], where the singularities of the bosonic two-point amplitudes were studied, after a suitable off-shell continuation [17], in order to perform a comparison with the ones occurring in the corresponding noncommutative field amplitudes. In particular the appearance in these amplitudes of an unphysical cut when the noncommutativity parameter involves the time variable was related to the classical instability of the corresponding string theory in the presence of a background external field of an "electric" type.

The analysis above was generalized in [18, 19] to the four-point "on-shell" tachyonic amplitude, thereby avoiding any possible trouble with off-shell continuations.

In the following we start considering again the bosonic string case, namely we study four-string amplitudes in the presence of a background field, first at tree level and then at one loop level, paying particular attention at the spectrum of the various singularities and at the related factorization properties. Later we repeat the above analysis for superstring amplitudes, where unsatisfactory unphysical features are no longer present.

We consider the bosonic case first since, although affected by the well-known pathologies, it still exhibits in a simpler setting most of the features we shall later encounter in the better grounded superstring context.

The motivation for looking at tree level amplitudes is prompted by the desire of exploring the open string spectrum in the presence of the B-field; such a field entails the occurrence of extra poles which otherwise would decouple for spatial symmetry reasons. One-loop amplitudes are instead essential to capture effects from the closed string spectrum, in particular the open-closed string vertices which can be obtained via factorization. These are the amplitudes related to the non-planar ones in the NCFT limit.

In the last part of our work such results will be extended to the superstring case, where pathologies related to ghosts and to redundant dimensions are absent. Here a stack of N Dp-branes is to be introduced in order to deal with the U(N) group [20, 21] and the B-field is chosen with non-vanishing components only in the directions of the branes. Actually two stacks of branes will be considered at a relative distance $\vec{Y}$, with string exchanges in between them. Again the presence of the B-field will reduce part of existing symmetries and thereby allow the presence of further poles which would decouple otherwise. Consistency with factorization properties following from unitarity will be checked explicitly. In particular the two different situations, characterized by a field B of a "magnetic" and of an "electric" type respectively, are discussed and compared. In the first case the theory does not exhibit any perturbative instability, at least in the one-loop amplitude we have considered. In the electric case instead a classical instability appears in the perturbative string amplitude when the "electric" field overcomes a critical value [22, 23, 24], due to an uncontrolled growth of the oscillation of modes in the direction parallel to the field. Corresponding to this value a violation of unitarity occurs in the form of a cut in the complex squared energy plane of tachyonic type.

When the Seiberg-Witten limit is considered leading to an effective field theory of a noncommutative type, the electric field is necessarily pushed into the instability region and the resulting theory is sick.

In section 2 the bosonic case is studied, first at the tree level, pointing out the peculiarities related to the presence of the B-field, and then at one loop where the interplay between closed and open sectors becomes apparent. Section 3 is devoted to superstring amplitudes, where analogous features occur in the scattering of strings lying on two stacks of branes at a relative distance $\vec{Y}$. When eventually the Seiberg-Witten limit is performed, several peculiarities of noncommutative field theories find their *raison d'être* in the corresponding features of the parent string theory. Final comments are the content of section 4.

2. The bosonic case

2.1 Tree amplitudes

Before entering *in medias res*, some notations and definitions have to be recalled. The following expression

$$S_{\text{bos}} = \frac{1}{4\pi\alpha'} \int_C d^2 z \, (g_{\mu\nu} \partial_a X^\mu \partial^a X^\nu - 2i\pi\alpha' B_{ij} \epsilon^{ab} \partial_a X^i \partial_b X^j) \tag{2.1}$$

is the action of a bosonic open string attached to a D-brane lying in the first $p+1$ dimensions and coupled to an antisymmetric constant background [1, 7]. We denote by latin letters $i, j, \ldots$ the components along the brane. The open string parameters are:

$$G = (g - 2\pi\alpha' B) g^{-1} (g + 2\pi\alpha' B)\,, \tag{2.2}$$

$$\theta = -(2\pi\alpha')^2 (g + 2\pi\alpha' B)^{-1} B (g - 2\pi\alpha' B)^{-1}\,. \tag{2.3}$$

Here G_{ij} and g_{ij} are the open and the closed-string metric tensors, respectively.

We consider for simplicity the four-tachyon tree amplitude and introduce the usual Mandelstam variables $s = -(k_1+k_2)^2$, $t = -(k_1+k_4)^2$ and $u = -(k_1+k_3)^2$ using the metric tensor G_{ij}. The tachyons are on shell, $k_i^2 = -m^2$, $i = 1, \ldots, 4$. Our open string metric is $(-1, 1, \ldots, 1)$ and we choose our units so that $m^2 = -2$, $(\alpha' = 1/2)$. As a consequence $s + t + u = -8$.

The presence of the B-field affects the familiar Veneziano expression by a phase factor

$$\begin{aligned} A(k_1, \ldots, k_4) = {} & \frac{\Gamma(-1-s/2)\Gamma(-1-t/2)}{\Gamma(-2-s/2-t/2)} \exp\left(\frac{-i}{2}(k_2\tilde{k}_1 + k_3\tilde{k}_1 + k_3\tilde{k}_2)\right) + \\ & + \text{non-cyclic permutations}, \end{aligned} \tag{2.4}$$

where $(\tilde{k})^i = \theta^{ij}(k)_j$ and $\theta^{ij} = \pi\left(\frac{1}{g+\pi B}\right)^{ij}_A$. $(\ )_A$ denotes the antisymmetric part of a matrix.

No Chan-Paton factors are considered for the time being; we restrict ourselves to the U(1) case. This is not a real limitation and will be removed when considering one-loop amplitudes.

We are now interested in studying the pole of eq. (2.4) at $s = 0$. One can easily realize that the relevant terms are

$$\begin{aligned} A(k_1, \ldots, k_4)|_{\text{rel}} = {} & \left[\frac{\Gamma(-1-s/2)\Gamma(-1-t/2)}{\Gamma(-2-s/2-t/2)} \exp\left(\frac{-i}{2}(k_2\tilde{k}_1 + k_3\tilde{k}_1 + k_3\tilde{k}_2)\right) + \right. \\ & \left. + (k_1 \leftrightarrow k_2)\right] + (k_3 \leftrightarrow k_4)\,. \end{aligned} \tag{2.5}$$

The amplitude on the pole behaves like

$$
\begin{aligned}
A(k_1,\ldots,k_4) \;\simeq\; & -\frac{2}{s}\exp\left(\frac{-i}{2}k_3(\tilde{k}_1+\tilde{k}_2)\right)\left[\left(2+\frac{t}{2}\right)\exp\left(\frac{-i}{2}k_2\tilde{k}_1\right)+\right.\\
& \left.+\left(2+\frac{u}{2}\right)\exp\left(\frac{i}{2}k_2\tilde{k}_1\right)\right]\\
+(k_3\leftrightarrow k_4) \;=\; & \frac{2}{s}(t-u)\sin\frac{k_1\tilde{k}_2}{2}\sin\frac{k_3\tilde{k}_4}{2},
\end{aligned}
\tag{2.6}
$$

where momentum conservation and the mass shell conditions have been taken into account.

It is immediately clear that, in the absence of the B-field ($\theta^{ij}=0$), the above residue vanishes: The two tachyons cannot couple to a photon while respecting Bose statistics. In the presence of the B-field a nice factorization occurs of the residue as the product of two vertices, each carrying a Moyal phase, as expected from the result in ref. [1]. Indeed the vertex tachyon-tachyon-photon takes the form

$$
V^i(k_1,k_2)\simeq(k_1^i-k_2^i)\sin\frac{k_1\tilde{k}_2}{2}\,, \tag{2.7}
$$

and the presence of the Moyal phase is crucial to comply with Bose statistics. We notice that the transversality condition $(k_1+k_2)\cdot V=0$ is satisfied.

2.2 One-loop amplitudes

We turn now our attention to the study of one-loop amplitudes, always in the presence of a constant field B. In so doing our purpose is to explore the features of closed string poles in addition to the open string ones.

At one loop the string world-sheet is the cylinder $C_2=\{0\le \mathrm{Re}\,w\le 1, w=w+2i\tau\}$.

The one loop propagator with the boundary conditions imposed by the B-term, can be found in [7]. If one sets $w=x+iy$, the relevant propagator on the boundary of the cylinder ($x=0,1$) can be written as

$$
G(y,y') = \frac{1}{2}\alpha' g^{-1}\log q - 2\alpha' G^{-1}\log\left[\frac{q^{1/4}}{D(\tau)}\vartheta_4\left(\frac{|y-y'|}{2\tau},\frac{i}{\tau}\right)\right], \qquad x\neq x'\,, \tag{2.8}
$$

$$
G(y,y') = \frac{\pm i\theta}{2}\epsilon_\perp(y-y') - 2\alpha' G^{-1}\log\left[\frac{1}{D(\tau)}\vartheta_1\left(\frac{|y-y'|}{2\tau},\frac{i}{\tau}\right)\right], \qquad x=x'\,, \tag{2.9}
$$

where $q=e^{-\pi/\tau}$, $\pm$ correspond to $x=1$ and $x=0$ respectively, and $\epsilon_\perp(y)=\mathrm{sign}(y)-y/\tau$.

Here $\vartheta_{1,4}(\nu,\tau)$ are Jacobi theta functions, while $D(\tau)=\tau^{-1}[\eta(i/\tau)]^3$ and η is the Dedekind eta function [25].

With this propagator and the suitable modular measure, the amplitude for the insertion of N tachyonic vertex operators at $x=1$ and $M-N$ at $x=0$ turns out to be [7]:

$$
\begin{aligned}
A_{1,\ldots,M} = \mathcal{N}_0 & \int_0^\infty \frac{d\tau}{\tau}\tau^{-\frac{d}{2}}[\eta(i\tau)]^{2-d}q^{\frac{1}{2}\alpha' K g^{-1}K}\times\\
& \times\int[dy]\prod_{i=1}^{N}\prod_{j=N+1}^{M}\left[\frac{q^{\frac{1}{4}}\,\vartheta_4(\frac{|y_i-y_j|}{2\tau},\frac{i}{\tau})}{D(\tau)}\right]^{2\alpha' k_i G^{-1}k_j}\times
\end{aligned}
$$

$$\times \prod_{1=i<j}^{N} e^{-\frac{1}{2}i\epsilon_\perp(y_i-y_j)k_i\theta k_j}\left[\frac{\vartheta_1(\frac{|y_i-y_j|}{2\tau},\frac{i}{\tau})}{D(\tau)}\right]^{2\alpha' k_i G^{-1}k_j} \times$$
$$\times \prod_{N+1=i<j}^{M} e^{\frac{1}{2}i\epsilon_\perp(y_i-y_j)k_i\theta k_j}\left[\frac{\vartheta_1(\frac{|y_i-y_j|}{2\tau},\frac{i}{\tau})}{D(\tau)}\right]^{2\alpha' k_i G^{-1}k_j}. \tag{2.10}$$

Here $\mathcal{N}_0$ is a normalization constant, $d = p+1$, and $K = \sum_{i=1}^{N} k_i$ is the sum of all momenta associated with the vertex operators inserted on the $x = 1$ boundary. The integration region $[dy]$ for the variables y_i will be specified later on.

Whenever $N \neq 0$ and $M > N$, this amplitude corresponds to non-planar graphs, the traces of the relevant Chan-Paton matrices being understood.

For electric backgrounds it is well known that problems arise when the electric field approaches a critical value E_{cr}. Beyond it a classical instability occurs [22, 23, 24] both for neutral (which is the case we consider here) and for charged open strings, related to an uncontrolled growth of the oscillation amplitude of modes in the direction parallel to the field.

This phenomenon coexists with the quantum instability of purely charged strings due to pair production in any electric field (even in a sub-critical one), which is the analog of the Schwinger phenomenon in particle electrodynamics. However, at variance with the latter, it has no analog in particle field theory.[1] We will discuss it in subsection 3.2.

2.3 Closed string poles

We write the one-loop non-planar amplitude for a scattering of four bosonic open string tachyons in $d = 26$ (D25-brane) in the presence of a constant antisymmetric background as follows:

$$\mathcal{A} = A(1,2,3,4)\,\mathrm{Tr}[\lambda_1\lambda_2]\,\mathrm{Tr}[\lambda_3\lambda_4] + \text{non-trivial permutations}\,, \tag{2.11}$$

where the amplitude $A(1,2,3,4)$ above, after specializing eq. (2.10) (see appendix A), takes the form

$$A = \mathcal{N}\int_0^1 \frac{dq}{q^3}[f(q^2)]^{-16} q^{\frac{1}{4}Kg^{-1}K}\int_0^1 d\nu_1\int_0^{\nu_1} d\nu_2\int_0^{\nu_2} d\nu_3 e^{-\frac{i}{2}[k_1\theta k_2(1-2\nu_{12})-k_3\theta k_4(1-2\nu_3)]}\times$$
$$\times\left[\left[\sin\pi\nu_{12}\prod_1^\infty\left(1-2q^{2n}\cos 2\pi\nu_{12}+q^{4n}\right)\right]\left[\sin\pi\nu_3\prod_1^\infty\left(1-2q^{2n}\cos 2\pi\nu_3+q^{4n}\right)\right]\right]^{-2-(s/2)}\times$$
$$\times\left[\prod_1^\infty\left(1-2q^{2n-1}\cos 2\pi\nu_{13}+q^{4n-2}\right)\right]^{-2-(u/2)}\left[\prod_1^\infty\left(1-2q^{2n-1}\cos 2\pi\nu_1+q^{4n-2}\right)\right]^{-2-(t/2)\times}$$
$$\times\left[\prod_1^\infty\left(1-2q^{2n-1}\cos 2\pi\nu_{23}+q^{4n-2}\right)\right]^{-2-(t/2)}\left[\prod_1^\infty\left(1-2q^{2n-1}\cos 2\pi\nu_2+q^{4n-2}\right)\right]^{-2-(u/2)} \tag{2.12}$$

[1] In passing, we recall that for neutral strings (q-charge on one end, $-q$ on the other), one has $|E_{cr}| = 1/(2\pi\alpha'|q|)$. For charges $q_1 \neq q_2$ on the two boundaries, one finds that the pair production rate diverges at a critical value $E_{cr} = 1/(2\pi\alpha'|max\, q_i|)$.

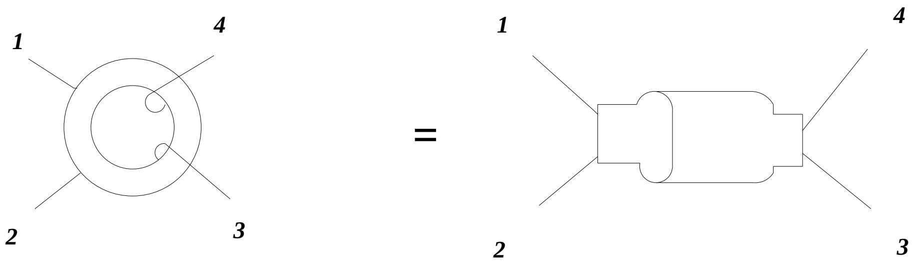

Figure 1: Diagram representing the amplitude $A(1,2,3,4)$.

Eq. (2.12) is the amplitude for a fixed order of the external momenta (see the diagram in figure 1); the coordinates ν_i of the vertex operator insertions are taken to be cyclically ordered from zero to one on each of the two boundaries, with ν_4 fixed to zero in order to remove the residual gauge. All possible non-trivial permutations of the labels of the external particles are to be summed over. The ones concerning k_3 are accounted for by extending the region of integration over ν_3 from 0 to 1. Henceforth we shall denote by the symbol $A(1,2)$ such an extension $(0 \leq \nu_3 \leq 1)$. We have expressed the amplitude using the variable $q = \exp[-\pi/\tau]$, which is the traditional one that reference [26] uses for the spectral analysis in the case without B-field, in order to have a direct comparison. External momenta are taken to be on the tachyon mass-shell. Indices are raised and lowered with the open string metric G^{-1}. We define $\nu_{rs} = \nu_r - \nu_s$; $K = k_1 + k_2$ is the total momentum entering through one boundary and leaving from the other. We also define $f(x) = \prod_{n=1}^{\infty}(1 - x^n)$ as in [26].

It is well known that, in the case without the B-field, this amplitude presents a rich pattern of singularities, each of them being required by unitarity, corresponding to precise intermediate exchanges.

First of all, if one takes the amplitude $A(1,2)$ as an example, and analyzes it in the variable s, setting $\theta = 0$ and $g = G$ inside the integral, one finds a series of simple poles when s equals the masses of the closed string tower. This is required by unitarity, because a diagram like the one of figure 1 is topologically equivalent to a tree-level closed string exchange between the sets of particles $\{1,2\}$ and $\{3,4\}$.

These poles emerge when one explores the small region of the integration over the q-variable; in particular, one easily sees that the amplitude can be written as follows

$$A(1,2) = \sum_{n=0}^{\infty} \int_0^1 dq\, q^{-3+n-(s/4)}\; a_n(s,t,u), \tag{2.13}$$

namely

$$A(1,2) = \sum_{n=0}^{\infty} \frac{1}{(-2+n-(s/4))}\, a_n(s,t,u). \tag{2.14}$$

From this expansion one might naively conclude that poles occur at the masses $s = 4n-8 = -8, -4, 0, \ldots$, which, unless canceled by the sum over permutations, would be at odds with unitarity, since the expected closed string spectrum is $s = -8, 0, \ldots$ Actually quantities like $a_1, a_3, \ldots$ cancel on their own, as one can check explicitly by performing the integrals over the ν_i or as a particular case of appendix B.

The first thing to be checked is whether such a cancellation persists in the presence of the B-field; we exhibit it here explicitly for the lowest level $n = 1$. The remarkable proof of cancellation for higher odd values of n is given in appendix B. It is easy to realize from eq. (2.12) that, even in the presence of the B-field, a formula analogous to (2.14) can be written

$$A(1,2) = \sum_{n=0}^{\infty} \frac{1}{(-2+n-(s_{CL}/4))}\, \alpha_n(s,t,u,k_1\theta k_2, k_3\theta k_4)\,. \tag{2.15}$$

We have defined $s_{CL} = -Kg^{-1}K$ the squared energy variable in the closed channel, which is different from the one in the open channel s as long as B is different from zero.

We perform a Taylor expansion of the integrand in the expression of A up the first order in q and get

$$\begin{aligned}\alpha_1 &= \mathcal{N}\int_0^1 d\nu_1 \int_0^{\nu_1} d\nu_2 \int_0^1 d\nu_3 e^{-\frac{i}{2}k_1\theta k_2(1-2\nu_{12})} e^{\frac{i}{2}k_3\theta k_4(1-2\nu_3)} (\sin\pi\nu_{12}\sin\pi\nu_3)^{-2-(s/2)} \times \\ &\quad \times [(4+u)(\cos 2\pi\nu_{13}+\cos 2\pi\nu_2) + (4+t)(\cos 2\pi\nu_1 + \cos 2\pi\nu_{23})]\,. \end{aligned}\tag{2.16}$$

We now change variables to $p = \nu_{12}$; the integration region becomes $\int_0^1 dp \int_p^1 d\nu_1 \int_0^1 d\nu_3$. We easily integrate over ν_1 and, after use of trigonometric identities, obtain

$$\begin{aligned}\alpha_1 &= \frac{-4\mathcal{N}}{2\pi}\int_0^1 dp \int_0^1 d\nu_3 e^{-\frac{i}{2}k_1\theta k_2(1-2p)} e^{\frac{i}{2}k_3\theta k_4(1-2\nu_3)} (\sin\pi p \sin\pi\nu_3)^{-2-(s/2)} \times \\ &\quad \times \left[(8+t+u)(\sin\pi p\cos\pi p - \sin\pi p\cos\pi p\sin^2\pi\nu_3) + (u-t)\sin\pi\nu_3\cos\pi\nu_3\sin^2\pi p\right]. \end{aligned}\tag{2.17}$$

Using the formulas (B.16), we finally get

$$\begin{aligned}\alpha_1 = \frac{2^s i\mathcal{N}}{\pi^2}\Bigg[& \frac{4(8+t+u)\,\Gamma(-1-\frac{s}{2})\Gamma(-\frac{s}{2})\,k_1\theta k_2}{\Gamma(-\frac{s}{4}+\frac{k_3\theta k_4}{2\pi})\Gamma(-\frac{s}{4}-\frac{k_3\theta k_4}{2\pi})\Gamma(1-\frac{s}{4}+\frac{k_1\theta k_2}{2\pi})\Gamma(1-\frac{s}{4}-\frac{k_1\theta k_2}{2\pi})} - \\ & - \frac{(8+t+u)\,\Gamma(1-\frac{s}{2})\Gamma(-\frac{s}{2})\,k_1\theta k_2}{\Gamma(1-\frac{s}{4}+\frac{k_3\theta k_4}{2\pi})\Gamma(1-\frac{s}{4}-\frac{k_3\theta k_4}{2\pi})\Gamma(1-\frac{s}{4}+\frac{k_1\theta k_2}{2\pi})\Gamma(1-\frac{s}{4}-\frac{k_1\theta k_2}{2\pi})} - \\ & - \frac{(u-t)\,\Gamma(1-\frac{s}{2})\Gamma(-\frac{s}{2})\,k_3\theta k_4}{\Gamma(1-\frac{s}{4}+\frac{k_3\theta k_4}{2\pi})\Gamma(1-\frac{s}{4}-\frac{k_3\theta k_4}{2\pi})\Gamma(1-\frac{s}{4}+\frac{k_1\theta k_2}{2\pi})\Gamma(1-\frac{s}{4}-\frac{k_1\theta k_2}{2\pi})}\Bigg]. \end{aligned}\tag{2.18}$$

The first thing to notice is that, in the limit $\theta \to 0$, all this expression vanishes and we recover the mentioned result when $B = 0$. On the contrary, for a non-vanishing θ, (2.18) is different from zero. As already mentioned, this would contradict unitarity, since the pattern of the closed string spectrum is unaffected by the presence of B. Now the cancellation is subtler; we notice that the expression (2.18) is odd under the exchange $k_1 \leftrightarrow k_2$, as θ is antisymmetric and $t \leftrightarrow u$ under such an exchange (the Chan-Paton factors in (2.11) remain the same). Therefore, the sum over the other three diagrams considered in (2.11) gets rid of such an unwanted singularity.

The closed poles in the non-planar diagram of figure 1 correspond to the traditional pattern. To get the cancellation of the unwanted closed poles one needs to sum over permutations: The expected structure is recovered only at the level of the complete four-point amplitude.[2]

[2] This could also be interpreted as an indication of the need to sum over all possible configurations in dealing with amplitudes in the presence of the B-background.

2.4 Open string poles

The amplitude $A(1,2,3,4)$ in the case of $B = 0$ (or, equivalently, $\theta = 0$) presents as well a series of open string poles in the s-channel with singlet quantum numbers. They arise when the ν_i's on the same boundary of the diagram in figure 1 get close to a common value.

When $B \neq 0$, $s_{CL} \neq s$ and there are two variables in which singularities may occur. To simplify our analysis, we single out the first non-vanishing residue in the variable s_{CL}, α_0, and examine its behaviour as a function of s. The very same Taylor expansion of the integrand of $A(1,2)$ in the variable q reveals that α_0 is given by

$$\alpha_0 = \mathcal{N} \int_0^1 dp \int_0^1 d\nu_3 e^{-\frac{i}{2}k_1\theta k_2(2p-1)} e^{\frac{i}{2}k_3\theta k_4(1-2\nu_3)}\, p \left(\sin \pi p \sin \pi \nu_3\right)^{-2-(s/2)} \tag{2.19}$$

and, using formulas easily derived from (B.16), we find

$$\alpha_0 = \frac{-i\mathcal{N} 2^{3+s}\, \Gamma^2(-1-\frac{s}{2}) \left[\pi i + \psi(-\frac{s}{4} + \frac{k_1\theta k_2}{2\pi}) - \psi(-\frac{s}{4} - \frac{k_1\theta k_2}{2\pi})\right]}{\pi \Gamma(-\frac{s}{4} + \frac{k_3\theta k_4}{2\pi})\Gamma(-\frac{s}{4} - \frac{k_3\theta k_4}{2\pi})\Gamma(-\frac{s}{4} + \frac{k_1\theta k_2}{2\pi})\Gamma(-\frac{s}{4} - \frac{k_1\theta k_2}{2\pi})}, \tag{2.20}$$

with $\psi(z) = \Gamma'(z)/\Gamma(z)$.

As a first remark, we notice that, when θ goes to zero, this result smoothly tends to the expression

$$a_0 = \frac{\mathcal{N}\, \Gamma^2(-\frac{2+s}{4})}{2\pi \Gamma^2(-\frac{s}{4})} \tag{2.21}$$

of eq. (2.14), which exhibits double poles at $s = -2, 2, 6, \ldots$

After the symmetrization $k_1 \leftrightarrow k_2$, the terms containing the ψ-function cancel and we are left with

$$\alpha_0(1,2) + \alpha_0(2,1) = \frac{\mathcal{N} 2^{4+s}\, \Gamma^2(-1-\frac{s}{2})}{\Gamma(-\frac{s}{4} + \frac{k_3\theta k_4}{2\pi})\Gamma(-\frac{s}{4} - \frac{k_3\theta k_4}{2\pi})\Gamma(-\frac{s}{4} + \frac{k_1\theta k_2}{2\pi})\Gamma(-\frac{s}{4} - \frac{k_1\theta k_2}{2\pi})}\,. \tag{2.22}$$

Now the poles, which were missing when $\theta = 0$, are switched on, the presence of the B-field providing extra structures. Indeed the gamma function in the numerator has poles at $s = -2, 0, 2, 4, \ldots$, namely at all the open masses. There is no cancellation from the denominator but for quite specific values of the external momenta such that $k_1\theta k_2$ or $k_3\theta k_4$ is zero.

These open string poles deserve some comments. As is well known, the form of the Chan-Paton factors of the non-planar amplitude (2.11) of figure 1 indicates that intermediate s-channel states are "colour"-singlets. In figure 2 we have drawn a field-theory like diagram representing the amplitude $A(1,2,3,4)$. If we single out the first contribution from the closed poles, namely the closed-tachyon exchange with residue α_0, we can represent it by a thick line. In addition, double poles arise from integration of the sin-functions, which produce singularities when the variables ν_{12} and ν_3 approach the values zero or one, the corresponding vertex insertions pinching as in figure 2. An open string tower travels along the two thin lines, but only singlet states can couple to the closed sector.

Without the B-field only a few groups are allowed, according to tree level unitarity: SO(n) and USp(n) for non-orientable open strings, U(n) for orientable ones. In the non-orientable case, levels $s = -2, 2, \ldots$ (open tachyon, open massive graviton, $\ldots$) contain singlets, while levels $s = 0, 4, \ldots$ (open vector, $\ldots$) do not. The pattern of double poles of (2.21) is thereby explained.

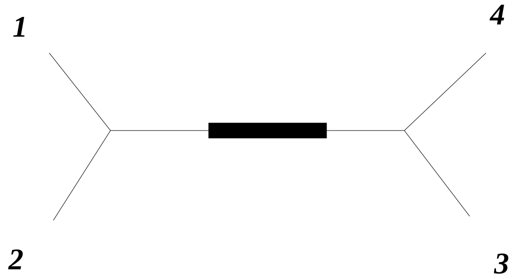

Figure 2: Field-theoretical representation of the amplitude $A(1,2,3,4)$.

In the orientable case the level $s = 0$ contains a singlet, which is the U(1) component in the splitting U(n) = SU$(n) \times$ U(1) and could couple in principle to the closed channel. Its absence in a_0 is explained if one realizes that the tree level vertex for the production of a U(1) photon from two tachyons vanishes for symmetry reasons (see subsection 2.1).

When the B-field is present, we find a different situation: first of all, the allowed gauge group undergoes a restriction [7], and we should consider only U(n). This is in agreement with the pattern of poles of (2.22), where the double open pole at $s = 0$ is now present. As a matter of fact the B-field does not alter the nature of the open string spectrum either: Since in the case of a gauge group SO(n) or USp(n) the mass-level $s = 0$ does not contain a singlet component, this would be inconsistent with our findings. But groups SO(n) and USp(n) are indeed forbidden.

For U(n), the U(1) part of the vector still contributes in the closed channel and we find the double open string pole at $s = 0$ in (2.22). This is due to a "non-decoupling" of the U(1) part of U(n), already encountered in the presence of the B-field for example in the vector amplitudes [17]. The simple reason for this phenomenon is that the above mentioned tree-level vertex for the production of a U(1) photon from two tachyons is now proportional to $\mathrm{Tr}[\lambda_1\lambda_2](p_1 - p_2)_\mu \exp[-ip_1\theta p_2]$, which does not vanish after symmetrization (see again subsection 2.1). The U(1) photon can therefore be produced as a singlet and couples to the closed channel.

In analogy with the factorization found in subsection 2.1, we can now consider the expression (2.22) and single out the residue at the double pole $s = 0$. Exploiting the properties of the Gamma functions, we get:

$$\begin{aligned}\alpha_0(1,2) + \alpha_0(2,1) \simeq\ & \mathrm{Tr}[\lambda_1\lambda_2]\,\mathrm{Tr}[\lambda_3\lambda_4]\frac{16}{\pi^4}\frac{\mathcal{N}}{s^2}\frac{1}{(-2 - s_{CL}/4)} \times \\ & \times (k_1\theta k_2)(k_3\theta k_4)\sin\left[\frac{k_1\theta k_2}{2}\right]\sin\left[\frac{k_3\theta k_4}{2}\right]\end{aligned} \tag{2.23}$$

where we have for clarity re-inserted the closed tachyon pole and the Chan-Paton factors. This part of the amplitude is completely factorisable in the following form:

$$\mathcal{N}_1\,\mathrm{Tr}[\lambda_1\lambda_2]\sin\left[\frac{k_1\theta k_2}{2}\right](k_1 - k_2)_i\,\frac{G^{ij}}{s}\,D_{jl}(K)\,\frac{G^{lm}}{s}(k_3 - k_4)_m\sin\left[\frac{k_3\theta k_4}{2}\right]\mathrm{Tr}[\lambda_3\lambda_4]\mathcal{N}_1\,, \tag{2.24}$$

where

$$D_{ij}(K) = \mathcal{N}_2\,\theta_{il}K^l\,\frac{1}{(-2 - s_{CL}/4)}\,\theta_{jm}K^m; \tag{2.25}$$

antisymmetry of θ and total momentum conservation ($K = k_1 + k_2 = -k_3 - k_4$) have been taken into account.

The factorization above can be easily traced back from figure 2. In order to complete the proof we should convince the reader that the "effective closed photon propagator" has indeed the expression of eq. (2.25). The one-loop non-planar off-shell vectorial 2-point amplitude on the brane with momentum K takes the form [17]

$$A^{ij} = \mathcal{N}_1 \int_0^\infty [d\tau] \int_0^1 d\nu\, e^{-\frac{\alpha'\pi}{2\tau}Kg^{-1}K} \left[\frac{e^{-\frac{\pi}{4\tau}}\vartheta_4(\nu, \frac{i}{\tau})}{(\frac{1}{\tau})[\eta(\frac{i}{\tau})]^3}\right]^{-2\alpha' KG^{-1}K} \times (\rho' J^{ij})\,, \tag{2.26}$$

where

$$[d\tau] = d\tau(\tau)^{1-d/2}\frac{1}{x}\prod_{n=1}^{\infty}(1-x^n)^{2-d}, \tag{2.27}$$

$$x = e^{-2\pi\tau} \tag{2.28}$$

$$\rho' = -e^{-2\pi\tau\nu} \tag{2.29}$$

$$J = J_0 + J_1 + J_2 \tag{2.30}$$

with

$$\begin{aligned} J_0 &= -2\alpha'\rho'\left(\frac{\partial I_0}{\partial\rho'}\Big|_{\rho=1}\right)^2 [G^{ij}K^2 - K^iK^j] \\ J_1 &= \frac{i}{\log x}\left(\tilde{K}^iK^j\frac{\partial I_0}{\partial\rho'} + K^i\tilde{K}^j\frac{1}{\rho'}\frac{\partial I_0}{\partial\rho}\right)\Big|_{\rho=1} \\ J_2 &= -\frac{1}{2\alpha'(\log x)^2}\frac{1}{\rho'}\tilde{K}^i\tilde{K}^j\,, \end{aligned} \tag{2.31}$$

and

$$I_0 = \frac{\log^2|\rho\rho'|}{2\log x} + \log\left(\sqrt{\frac{|\rho|}{|\rho'|}} + \sqrt{\frac{|\rho'|}{|\rho|}}\right) + \log\prod_{n=1}^{\infty}\frac{(1+x^n|\rho/\rho'|)(1+x^n|\rho'/\rho|)}{(1-x^n)^2}. \tag{2.32}$$

We again define

$$s = -K^2 = -KG^{-1}K \tag{2.33}$$

$$s_{CL} = -Kg^{-1}K \tag{2.34}$$

and set $\alpha' = 1/2$ and $d = 26$.

We look at the singularity at $s = 0$. It is easy to realize that only the term J_2 survives, since $K^2 = 0$ and $K\cdot(k_1 - k_2) = K\cdot(k_3 - k_4) = 0$. The effective term in the amplitude is therefrom

$$A_{\text{eff}} = \mathcal{N}_2\int_0^1 d\nu\int_0^\infty \frac{d\tau}{\tau^2}\,\tau^{-12}e^{\frac{\pi}{4\tau}s_{CL}}\left[e^{-\pi\tau/12}\prod_{m=1}^{\infty}(1-e^{-2\pi\tau m})\right]^{-24}\tilde{K}^i\tilde{K}^j$$

$$
\begin{aligned}
&= \mathcal{N}_2 \int_0^\infty \frac{d\tau}{\tau^2}\, \tau^{-12} e^{\frac{\pi}{4\tau} s_{CL}} \left[\eta(i\tau)\right]^{-24} \tilde{K}^i \tilde{K}^j \\
&= \mathcal{N}_2 \int_0^\infty \frac{d\tau}{\tau^2}\, e^{\frac{\pi}{4\tau} s_{CL}} \left[\eta\left(\frac{i}{\tau}\right)\right]^{-24} \tilde{K}^i \tilde{K}^j \\
&= \mathcal{N}_2 \sum_{n=0}^{\infty} b_n \int_0^\infty dT\, e^{-T\pi(-\frac{s_{CL}}{4}-2+2n)} \tilde{K}^i \tilde{K}^j \\
&= \sum_{n=0}^{\infty} a_n \frac{1}{-\frac{s_{CL}}{4} - 2 + 2n} \tilde{K}^i \tilde{K}^j,
\end{aligned}
\tag{2.35}
$$

where we have used the relation $\eta(l) = (-il)^{-1/2}\eta(-1/l)$ for the Dedekind eta-function and expressed the infinite product $\prod_{m=1}^{\infty}(1 - e^{\frac{-2\pi m}{\tau}})^{-24}$ as a series $\sum_{n=0}^{\infty} b_n e^{-2\pi n T}$, $T = 1/\tau$.

The lowest level ($n = 0$) exactly coincides with the expression (2.25).

3. Generalization to superstrings

3.1 The superstring amplitude

When considering the corresponding superstring case few novelties will emerge. The action (2.1) will be replaced by its supersymmetric extension

$$
S = S_{\rm bos} + S_{\rm ferm}\,, \tag{3.1}
$$

with

$$
S_{\rm ferm} = \frac{1}{4\pi\alpha'} \int_\Sigma d\sigma d\tau \left(g_{\mu\nu}\, \bar{\xi}^\mu \partial_a \gamma^a \xi^\nu - 2i\pi\alpha' B_{ij}\, \epsilon^{ab} \bar{\xi}^i \partial_a \gamma_b \xi^j \right). \tag{3.2}
$$

Here ξ is a two-dimensional Majorana spinor and γ_a are the usual two-dimensional gamma matrices.

We consider the amplitude for the non-planar scattering of four on-shell open string vectors, attached on two parallel Dp-branes, for instance D3-branes, in type-II superstring

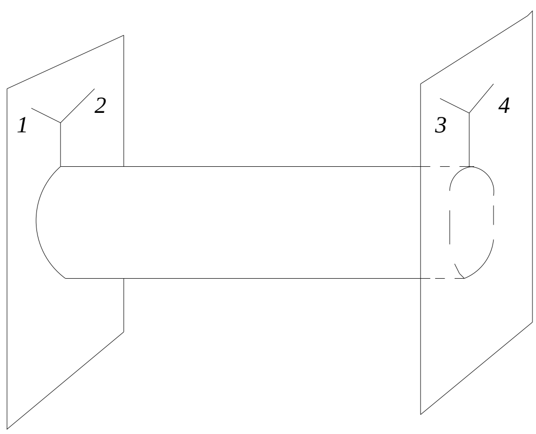

Figure 3: Diagram representing the amplitude $A_S(1,2,3,4)$.

theory [21]. Here $p < 9$ and one has type IIB for odd p, type IIA for even p. The diagram can be represented as in figure 3 and the amplitude can be written as

$$\mathcal{A}_{\mathcal{S}} = A_{S}(1,2)\,\mathrm{Tr}[\lambda_1\lambda_2]\,\mathrm{Tr}[\lambda_3\lambda_4] + \text{non trivial permutations} \tag{3.3}$$

where

$$\begin{aligned}
A_S(1,2) \,=\, & \mathcal{N}_{\mathcal{S}}\mathcal{K}\int_0^1 \frac{dq}{q} q^{\frac{1}{4}Kg^{-1}K}\log[q]^{(d/2)-5}\exp\left(\frac{\vec{Y}^2}{\log[q]}\right)\times \\
& \times\int_0^1 d\nu_1\int_0^{\nu_1} d\nu_2\int_0^1 d\nu_3 e^{-\frac{i}{2}k_1\theta k_2(1-2\nu_{12})}e^{\frac{i}{2}k_3\theta k_4(1-2\nu_3)}\times \\
& \times\left[\sin\pi\nu_{12}\prod_1^\infty(1-2q^{2n}\cos 2\pi\nu_{12}+q^{4n})\right]^{-(s/2)}\times \\
& \times\left[\sin\pi\nu_{3}\prod_1^\infty(1-2q^{2n}\cos 2\pi\nu_{3}+q^{4n})\right]^{-(s/2)}\times \\
& \times\left[\prod_1^\infty(1-2q^{2n-1}\cos 2\pi\nu_{13}+q^{4n-2})\right]^{-(u/2)}\times \\
& \times\left[\prod_1^\infty(1-2q^{2n-1}\cos 2\pi\nu_{1}+q^{4n-2})\right]^{-(t/2)}\times \\
& \times\left[\prod_1^\infty(1-2q^{2n-1}\cos 2\pi\nu_{23}+q^{4n-2})\right]^{-(t/2)}\times \\
& \times\left[\prod_1^\infty(1-2q^{2n-1}\cos 2\pi\nu_{2}+q^{4n-2})\right]^{-(u/2)}.
\end{aligned} \tag{3.4}$$

We follow the same notations of the previous sections. Moreover, $d = p+1$ is the dimension of the worldvolume of the branes, and $\vec{Y}$ is their spacelike separation as in [21, 20]. The external momenta all lie on the branes, which are infinitely heavy objects in perturbation theory, breaking translational invariance in the transverse space. We notice the absence of the function $f(q^2)$ when comparing the integrand with the corresponding one in the bosonic case, due to a cancellation between bosonic and fermionic sectors.

The new factor $\log[q]^{(d/2)-5}$ appears instead, which is absent in the critical dimension $d = 10$; a novelty is the term which accounts for the brane separation, which again would be absent for space-filling D9-branes.

However D9-branes are not possible here, for two reasons: The first one has to do with the fact that a D9-brane theory is the same as a type-I string theory, which is endowed with the SO(32) gauge group. We have already remarked that such a group is incompatible with the presence of the B-field [27]. The second one is that a D9-brane (type-I open string) has no NS-NS B-field in its closed sector, it has a R-R two form which does not couple to the fundamental superstring according to the simple expression (3.1).

A Dp-brane of type II has instead U(1) as natural gauge group (or U(n), if we have a stack of n overlapping branes). Such a group is allowed in the presence of the B-field: We have indeed a B-field from the closed sector of type II.

For the sake of generality we consider n_1 coincident branes, separated by a distance $|\vec{Y}|$ from a stack of other n_2 coincident branes of the same dimension $p+1$. Let the gauge algebra of λ_1, λ_2 be $u(n_1)$ and the one of λ_3, λ_4 $u(n_2)$.

$\mathcal{K}$ represents the tensorial structure of the amplitude: it originates from the trace over the fermionic part of the gauge boson vertex operator, saturated with the relevant polarization vectors [21]. One finds

$$\mathcal{K} = M_{1}{}_i^j M_{2}{}_j^l M_{3}{}_l^m M_{4}{}_m^n - \frac{1}{4} M_{1ij} M_2{}^{ji} M_{3lm} M_4{}^{ml} + 2\,\text{permutations}\,, \tag{3.5}$$

where $M_r{}^{ij} = e_r{}^i k_r{}^j - e_r{}^j k_r{}^i$, e_r being the polarization vectors; indices are raised and lowered with the open string metric.

The integrand has almost the same form of the bosonic case, apart from the different q-structure and the already mentioned changes. We can again write

$$A_S(1,2) = \sum_{n=0}^{\infty} f_n(s_{CL}, d, |\vec{Y}|)\, \alpha_n^S(s,t,u,k_1\theta k_2, k_3\theta k_4)\,, \tag{3.6}$$

with

$$f_n(s_{CL}, d, |\vec{Y}|) = \int_0^1 dq\, q^{-1+n-(s_{CL}/4)} \left(\log[q]\right)^{(d/2)-5} \exp\left(\frac{\vec{Y}^2}{\log[q]}\right). \tag{3.7}$$

The expression of α_1^S is essentially the same as the bosonic one eq. (2.18), with $\mathcal{N}_S$ replacing $\mathcal{N}$, the extra tensorial factor $\mathcal{K}$ and the shifts $(s,t,u) \to (s-4, t-4, u-4)$. In particular, it is again odd under the exchange of $k_1 \leftrightarrow k_2$ and therefrom vanishing when suitably symmetrized. According to appendix B only even terms survive

$$A_S(1,2) = \sum_{n=0}^{\infty} f_{2n}(s_{CL}, d, |\vec{Y}|)\, \alpha_{2n}^S(s,t,u,k_1\theta k_2, k_3\theta k_4)\,. \tag{3.8}$$

Let us give a closer look at the coefficients f_{2n}. We see that

$$f_{2n}(s_{CL}, d, |\vec{Y}|) = -4i \left(-\frac{1}{\pi}\right)^{\frac{d-10}{2}} G_{9-p}(\mu_{2n}^2;\, |\vec{Y}|)\,, \tag{3.9}$$

where $G_{9-p}(\mu_{2n}^2; |\vec{Y}|)$ is nothing but the propagator kernel for a particle of mass $\mu_{2n}^2 = 8n - s_{CL}$ at a spatial distance $|\vec{Y}|$

$$G_{9-p}(\mu_{2n}^2; |\vec{Y}|) = \frac{i}{(2\pi)^{9-p}} \int d^{9-p}\vec{k}\, \frac{\exp[i\vec{k}\cdot\vec{Y}]}{\vec{k}^2 + \mu_{2n}^2 - i\epsilon}\,. \tag{3.10}$$

If we recall that s_{CL} gets contribution from the momenta lying on the branes, and we split the momentum of a generic bulk particle as $K_{\rm tot}^2 = \vec{k}^2 - s_{CL}$, the above coordinate propagator can be thought as representing the Fourier transform of a momentum space propagator $\frac{1}{K_{\rm tot}^2+M^2-i\epsilon}$ for a spatial distance $|\vec{Y}|$ (the position of the two fixed branes). Then consistency requires $M^2 = 8n$, which is precisely the closed string spectrum. We stress once again that we need to suitably symmetrize the amplitude in order to recover the correct pattern of closed string poles.

The factorization in eq. (3.6) deserves a comment. The residues on the closed string poles possess a tensorial structure on their own, which depends on the level n we consider. When coupled to the branes, only the vector components along the branes survive and do not undergo the integration in (3.10). Their contribution is included in α_n^S.

Now we turn our attention to the contribution related to the massless closed pole, α_0^S: After symmetrization we obtain the same result as in eq. (2.22), with $s \to s-4$ and the factors $\mathcal{N}_{\mathcal{S}}$ and $\mathcal{K}$

$$\alpha_0^S(1,2)+\alpha_0^S(2,1) = \frac{\mathcal{N}_{\mathcal{S}}\mathcal{K}\, 2^s\, \Gamma^2(1-\frac{s}{2})}{\Gamma(1-\frac{s}{4}+\frac{k_3\theta k_4}{2\pi})\Gamma(1-\frac{s}{4}-\frac{k_3\theta k_4}{2\pi})\Gamma(1-\frac{s}{4}+\frac{k_1\theta k_2}{2\pi})\Gamma(1-\frac{s}{4}-\frac{k_1\theta k_2}{2\pi})}\,. \tag{3.11}$$

When θ vanishes, this expression becomes

$$a_0^S(1,2)+a_0^S(2,1) = \frac{\mathcal{N}_{\mathcal{S}}\mathcal{K}\,\Gamma^2(\frac{2-s}{4})}{\pi\Gamma^2(1-\frac{s}{4})}\,. \tag{3.12}$$

There are poles at $s = 2, 6, \ldots$, again in the correct traditional pattern.

When B is turned on, all the poles in the numerator of eq. (3.11), namely at $s = 2, 4, 6, 8, \ldots$ are present, the ones at $s = 4, 8, \ldots$ being no longer canceled by the denominator, apart from exceptional configurations of the momenta.

We notice a difference with the bosonic case: the open string pole at $s = 0$ is always absent. Without the B-field, this massless pole is forbidden by supersymmetry (the vanishing of the one loop three-vectors amplitude due to a non-renormalization of the three-vector vertex). The presence of B does preserve this constraint: In particular, any vector amplitude on the annulus with less than eight spinors, such as the three point amplitude, continues to be zero [21].

3.2 String singularities and unitarity in the Seiberg-Witten limit

We restore in the amplitude (3.4) the dependence on α' and introduce the notation $K \circ K \equiv -K\theta G\theta K$. The quantity $|\vec{Y}|/2\pi\alpha' \equiv m$ behaves like a mass scale in the theory; it is interpreted as the mass of the ground states of open strings stretching between the branes [21, 20]. We choose θ block-diagonal and distinguish between the "magnetic" case, where $K \circ K$ is positive definite (θ has only spatial components), and the "electric" case where, for simplicity, θ is chosen with the only non-vanishing component $\theta_E = \theta_{01}$.

We are now interested in exploring the analytic behaviour of the amplitude (3.4) as a function of the variable s at suitable fixed values of the other kinematic variables. This behaviour will be eventually compared with the one of the corresponding amplitude in the noncommutative field theory obtained by performing the Seiberg-Witten limit.

To this purpose we express the variable s_{CL} in term of s

$$s_{CL} = s - \frac{1}{4\pi^2\alpha'^2} K \circ K\,. \tag{3.13}$$

The amplitude has a rich pattern of singularities, due to the fact that it describes simultaneously several physical processes in different kinematic regions. Besides the double poles at $s = 2n$, $n = 1, 2, \ldots$ we have already mentioned, it exhibits a branch point at $s = 4m^2$ with a cut along the positive real axis. It is the unitarity cut related to the open strings attached to the branes. In the representation

$$A_S = \mathcal{N}_S \mathcal{K} \int_0^\infty \frac{dl}{l} l^{4-d/2} e^{-\frac{\pi^2\alpha'^2}{l} K g^{-1} K} e^{-\frac{\vec{Y}^2 l}{(2\pi\alpha')^2}} e^{-\frac{i}{2}k_1\theta k_2} e^{\frac{i}{2}k_3\theta k_4} \int_0^1 d\nu_1 \int_0^{\nu_1} d\nu_2 \times$$
$$\times \int_0^1 d\nu_3 e^{i(\nu_{12})k_1\theta k_2} e^{-i\nu_3 k_3\theta k_4} \prod_{n=1}^{2}\prod_{m=3}^{4} \left[\frac{e^{-\frac{\pi^2\alpha'}{2l}} \vartheta_4(\nu_{nm}, \frac{2\pi i\alpha'}{l})}{(\frac{2\pi\alpha'}{l})[\eta(\frac{2\pi i\alpha'}{l})]^3} \right]^{2\alpha' k_n G^{-1} k_m} \times$$
$$\times \left[\frac{\vartheta_1(\nu_{12}, \frac{2\pi i\alpha'}{l})}{(\frac{2\pi\alpha'}{l})[\eta(\frac{2\pi i\alpha'}{l})]^3} \right]^{2\alpha' k_1 G^{-1} k_2} \left[\frac{\vartheta_1(\nu_3, \frac{2\pi i\alpha'}{l})}{(\frac{2\pi\alpha'}{l})[\eta(\frac{2\pi i\alpha'}{l})]^3} \right]^{2\alpha' k_3 G^{-1} k_4}, \tag{3.14}$$

it originates from the integration over high values of l as explained in appendix C.

Low values of l describe instead the effect of the closed string exchange between the branes. The ensuing singularities are particularly interesting in the Seiberg-Witten limit. Eq. (3.7) leads to the following representation

$$f_n(s_{CL}, d, |\vec{Y}|) \propto \left(\frac{2\pi m\alpha'}{\sqrt{n - \alpha' s_{CL}/4}} \right)^{d/2-4} K_{d/2-4}\left(4\pi\alpha' m \sqrt{n - \frac{\alpha' s_{CL}}{4}} \right), \qquad n = 0, 1, \ldots \tag{3.15}$$

A branch point occurs when the argument of the square root becomes negative. If we trade the variable s_{CL} for s we obtain the conditions

$$s \geq \frac{4n}{\alpha'} + \frac{1}{4\pi^2\alpha'^2} K \circ K\,, \qquad n = 0, 1, \ldots \tag{3.16}$$

If θ is magnetic, $K \circ K$ is positive definite. We find thereby cuts along the positive real axis.

If instead θ is electric,

$$K \circ K = \theta_E^2 (K_0^2 - K_1^2) = \theta_E^2 s + \theta_E^2 K_T^2\,, \tag{3.17}$$

where $K_T^2 \equiv K_2^2 + K_3^2 + \cdots + K_{d-1}^2$. Eq. (3.16) then becomes

$$s_n \left(1 - \frac{\theta_E^2}{4\pi^2\alpha'^2} \right) = \frac{\theta_E^2}{4\pi^2\alpha'^2} K_T^2 + \frac{4n}{\alpha'}\,, \tag{3.18}$$

that, if we define $\tilde{E} = E/E_{cr} = \theta_E/2\pi\alpha'$, can be rewritten as

$$s_n(1 - \tilde{E}^2) = \tilde{E}^2 K_T^2 + \frac{4n}{\alpha'}\,. \tag{3.19}$$

Thereby the string theory is stable whenever $|\tilde{E}| < 1$, namely

$$s > \frac{\tilde{E}^2}{1 - \tilde{E}^2} K_T^2 + \frac{1}{1 - \tilde{E}^2} \frac{4n}{\alpha'} > 0\,. \tag{3.20}$$

When E overcomes E_{cr}, the theory exhibits a perturbative instability of a tachyonic type; the vacuum is likely to decay into a suitable configuration of branes.

We turn now our attention to the behaviour of such an amplitude in the Seiberg-Witten limit. It is well known that open strings in presence of an antisymmetric constant background are effectively described at low energy by noncommutative field theories. If in the amplitude (3.4) we perform the limit $\alpha' \to 0, g_{ij} \simeq \alpha'^2$, keeping G, θ and $m = |\vec{Y}|/2\pi\alpha'$ fixed, and suitably rescaling the string coupling constant (see [7, 1]), we get [21]

$$A = \mathcal{K} N \delta^d \left(\sum_{m=1}^{4} k_m \right) \int_0^\infty \frac{dl}{l} l^{4-\frac{d}{2}} e^{-\frac{1}{4l} K \circ K - m^2 l} e^{-\frac{i}{2} k_1 \theta k_2} e^{\frac{i}{2} k_3 \theta k_4} \times$$
$$\times \int_0^1 d\nu_1 \int_0^{\nu_1} d\nu_2 \int_0^1 d\nu_3 e^{-ik_1\theta k_2 \nu_{12}} e^{ik_3\theta k_4 \nu_3} \prod_{i<j=1}^{4} e^{l\nu_{ij}(1-\nu_{ij}) k_i G^{-1} k_j} \,. \qquad (3.21)$$

This expression corresponds to the sum of three non-planar diagrams.

One can easily check that in such a limit the open string (double) poles decouple. They are indeed absent in eq. (3.21). The right-hand cut for $s \geq 4m^2$, which is present (with a different discontinuity) also in commutative field theories due to intermediate on shell states, obviously survives.

A different fate occurs to the branch points related to the closed string singularities. In the magnetic case, if we look at the eq. (3.16), we see that in the limit $\alpha' \to 0$, they decouple as well.

If instead θ is of "electric" type, the Seiberg-Witten limit $\alpha' \to 0, \theta_E$ fixed, forces $\tilde{E} \to \infty > 1$. Even before reaching the noncommutative field theory limit, the string is pushed into its region of instability owing to the flipping of the branch point (see eq. (3.19)). In the meanwhile the closed branch points s_n get closer and closer to $-K_T^2$ (see eq. (3.18)). No wonder then that the amplitude in the resulting noncommutative theory eventually exhibits un unphysical cut of tachyonic type for $s < -K_T^2$.

4. Concluding remarks

The main purpose of this work was to explore up to one loop the perturbative spectra and the related factorization properties of (super)string amplitudes in the presence of a constant background field B. The final goal was to match peculiar (and sometimes pathological) features of noncommutative field theories, derived by means of the Seiberg-Witten limit, with the corresponding characteristics of the parent theory. This analysis was first performed in a simpler bosonic string context, in spite of the well-known difficulties due to the presence of tachyons and to the large number of extra dimensions.

Closed and open string poles occurring in the four open-string scattering amplitude were analyzed. After the required symmetrization, the closed string spectrum turns out to be unaffected by the presence of B, as expected from unitarity. Extra open string poles are instead present when B is switched on, this field providing further possible structures and therefrom a partial symmetry loss. Remarkable factorization properties occurring in the residues on the poles were found in full agreement with expected patterns.

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This analysis was then extended to the more satisfactory superstring context. Here two stacks of N (lower dimensional) branes at a relative distance $\vec{Y}$ were considered in order to deal with the Chan-Paton group $\mathrm{U}(N)$. This distance (divided by the string slope α') provides us with a mass scale, which can be interpreted as an IR or an UV effect, according to the different interpretations of figure 3 [20].

When the background field is of an "electric" type, there is a critical value of the electric field beyond which the string undergoes a classical perturbative instability and starts developing tachyonic poles. In the Seiberg-Witten limit this "electric" string is necessarily pushed into its instability region; the resulting noncommutative field theory turns out to exhibit an unphysical cut whose presence was noticed since a long time [2, 4].

A future step would be to explore this phenomenon in the more ambitious context of string field theories; this might allow to go beyond on-shell amplitudes and perhaps also beyond perturbation theory in the search of new, more satisfactory ground states.

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A. The four-point amplitude in the q-variable

We report here the relevant formulas which connect the τ-variable representation (2.10) and the q-variable representation (2.12) for the four-point amplitude.

Starting from the (2.10), we first perform the rescaling $t = 2\pi\alpha'\tau$ and $\nu_i = y_i/2\tau$. As mentioned in the text, we work in the gauge $\nu_4 = 0$. After such a rescaling, in (2.10) q becomes equal to $\exp[-2\pi^2\alpha'/t]$.

We use the well-known expansions of the Jacobi θ-functions [26]:

$$\theta_1\left[\nu_{ij}, \frac{2\pi i\alpha'}{t}\right] = 2\, f(q^2)\, q^{1/4} \sin\pi\nu_{ij} \prod_{n=1}^{\infty}(1 - 2q^{2n}\cos 2\pi\nu_{ij} + q^{4n})\,, \tag{A.1}$$

$$\theta_4\left[\nu_{ij}, \frac{2\pi i\alpha'}{t}\right] = f(q^2) \prod_{n=1}^{\infty}(1 - 2q^{2n-1}\cos 2\pi\nu_{ij} + q^{4n-2})\,, \tag{A.2}$$

where, as in the main text $\nu_{ij} = \nu_i - \nu_j$.

With the definition $\eta(s) = x^{\frac{1}{24}} \prod_{m=1}^{\infty}(1 - x^m)$ with $x = \exp[2\pi i s]$, we can write

$$D\left[\frac{t}{2\pi\alpha'}\right] = \frac{-\log q}{\pi}\, q^{1/4}\, f^3(q^2)\,, \tag{A.3}$$

where $f(q^2) = \prod_{n=1}^{\infty}(1 - q^{2n})$.

The duality transformation $\eta(s) = (-is)^{-1/2}\eta(-1/s)$ leads to

$$\eta\left(\frac{it}{2\pi\alpha'}\right) = \left(\frac{-\pi}{\log q}\right)^{-1/2} q^{1/12}\, f(q^2)\,. \tag{A.4}$$

Finally, changing variable from t to q and going on-shell in $d = 26$ (which in particular implies $\sum_{i<j} k_i k_j = -4$ if $\alpha' = 1/2$ and G is the Minkowski metric as in our conventions), we obtain eq. (2.12).

We stress that, like in the case without the B-field, only in the critical dimension and going "on-shell", the logarithms occurring in the measure exactly cancel against the ones coming from the propagator insertions, so that eventually only closed string poles ensue, in compliance with unitarity as explained in the text.

B. Cancellation of poles for odd values of n

The nth-order coefficient of the sum (2.15) is

$$\begin{aligned}\sum_h^n \sum_{\alpha_1+\alpha_2+\alpha_3+\alpha_4=h} \int_0^1 d\nu_3 \int_0^1 dp \int_p^1 d\nu_1 e^{-\frac{i}{2}k_1\theta k_2} e^{\frac{i}{2}k_3\theta k_4} = \\ = e^{ik_1\theta k_2\, p} e^{-ik_3\theta k_4\, \nu_3} A_h(\sin\pi p, \sin\pi\nu_3; s,t,u) \times \\ \times f_1(u)[\cos 2\pi(\nu_1-\nu_3)]^{\alpha_1}\, f_2(t)[\cos 2\pi\nu_1]^{\alpha_2} \times \\ \times f_3(t)[\cos 2\pi(\nu_1-p-\nu_3)]^{\alpha_3}\, f_4(u)[\cos 2\pi(\nu_1-p)]^{\alpha_4}\,,\end{aligned} \tag{B.1}$$

where h runs over values with the same parity of n and α_i are non-negative integer numbers whose dependence in f_a, $a = 1,\ldots,4$ is understood; A_h is a symmetric function of t and u.

This sum can be rewritten by symmetrizing over the variables α_i:

$$\begin{aligned}\frac{1}{8}\sum_h^n \sum_{\alpha_1+\alpha_2+\alpha_3+\alpha_4=h} \int_0^1 d\nu_3 \int_0^1 dp \int_p^1 d\nu_1 e^{-\frac{i}{2}k_1\theta k_2} e^{\frac{i}{2}k_3\theta k_4} \times \\ \times e^{ik_1\theta k_2\, p} e^{-ik_3\theta k_4\, \nu_3} A_h(\sin\pi p, \sin\pi\nu_3; s,t,u)\, [f_a(u) f_b(t)\mathcal{M}_1 + f_a(t) f_b(u)\mathcal{M}_2]\,,\end{aligned} \tag{B.2}$$

where

$$\begin{aligned}\mathcal{M}_1 = & \Big([\cos 2\pi(\nu_1-\nu_3)]^{\alpha_1}[\cos 2\pi(\nu_1-p)]^{\alpha_4} + [\cos 2\pi(\nu_1-\nu_3)]^{\alpha_4}[\cos 2\pi(\nu_1-p)]^{\alpha_1}\Big) \times \\ & \times\Big([\cos 2\pi(\nu_1-p-\nu_3)]^{\alpha_3}[\cos 2\pi\nu_1]^{\alpha_2} + [\cos 2\pi(\nu_1-p-\nu_3)]^{\alpha_2}[\cos 2\pi\nu_1]^{\alpha_3}\Big),\end{aligned} \tag{B.3}$$

$\mathcal{M}_2$ is obtained by performing the exchange $\alpha_1 \leftrightarrow \alpha_2$, $\alpha_4 \leftrightarrow \alpha_3$ in $\mathcal{M}_1$.

We want to verify that this expression vanishes when n is odd. It is enough to prove that

$$\begin{aligned}\int_0^1 d\nu_3 \int_0^1 dp \int_p^1 d\nu_1 e^{-\frac{i}{2}k_1\theta k_2} e^{\frac{i}{2}k_3\theta k_4} e^{ik_1\theta k_2\, p} e^{-ik_3\theta k_4\, \nu_3} \times \\ \times A_h(\sin\pi p, \sin\pi\nu_3; s,t,u)\, [f_a(u) f_b(t)\mathcal{M}_1 + f_a(t) f_b(u)\mathcal{M}_2] = 0\end{aligned} \tag{B.4}$$

for a generic choice of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ provided that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ is odd, after the symmetrization $k_1 \leftrightarrow k_2$.

Thanks to the symmetries $\alpha_1 \leftrightarrow \alpha_4$ and $\alpha_2 \leftrightarrow \alpha_3$, we can set $\alpha_1 < \alpha_4$ and $\alpha_2 < \alpha_3$ without loss of generality.

Let us consider $\mathcal{M}_1$. It can be written as

$$\begin{aligned}\mathcal{M}_1 = &[\cos 2\pi(\nu_1 - \nu_3)\cos 2\pi(\nu_1 - p)]^{\alpha_1} \cdot [(\cos 2\pi(\nu_1 - \nu_3))^{\alpha_4-\alpha_1} + (\cos 2\pi(\nu_1 - p))^{\alpha_4-\alpha_1}] \cdot \\ &\cdot[\cos 2\pi(\nu_1 - p - \nu_3)\cos 2\pi\nu_1]^{\alpha_2} \cdot [(\cos 2\pi(\nu_1 - p - \nu_3))^{\alpha_3-\alpha_2} + (\cos 2\pi\nu_1)^{\alpha_3-\alpha_2}], \end{aligned} \tag{B.5}$$

which, after using standard algebraic and trigonometric identities and dropping the irrelevant factor $2^{-\alpha_1-\alpha_2}$ common to $\mathcal{M}_1$ and $\mathcal{M}_2$, becomes

$$\begin{aligned}\mathcal{M}_1 = &[\cos 2\pi(2\nu_1 - p - \nu_3) + \cos 2\pi(p - \nu_3)]^{\alpha_1} \cdot [\cos 2\pi(2\nu_1 - p - \nu_3) + \cos 2\pi(p + \nu_3)]^{\alpha_2} \cdot \\ &\cdot\{[\cos \pi(2\nu_1 - p - \nu_3 - (p - \nu_3))]^{\alpha_4-\alpha_1} + [\cos \pi(2\nu_1 - p - \nu_3 + (p - \nu_3))]^{\alpha_4-\alpha_1}\} \cdot \\ &\cdot\{[\cos \pi(2\nu_1 - p - \nu_3 - (p+\nu_3))]^{\alpha_3-\alpha_2} + [\cos \pi(2\nu_1 - p - \nu_3 + (p+\nu_3))]^{\alpha_3-\alpha_2}\}. \end{aligned} \tag{B.6}$$

After some algebra, this quantity takes the form

$$\mathcal{M}_1 = \mathcal{F}_1 \cdot \mathcal{F}_2 \cdot \mathcal{F}_3 \cdot \mathcal{F}_4 \,, \tag{B.7}$$

with

$$\begin{aligned}\mathcal{F}_1 &= \sum_{n_1+m_1=\alpha_1} a_{n_1 m_1} \cos^{n_1} 2\pi(p - \nu_3)(1 - 2\sin^2 \pi(2\nu_1 - p - \nu_3))^{m_1} \,, \\ \mathcal{F}_2 &= \sum_{n_2+m_2=\alpha_2} a_{n_2 m_2} \cos^{n_2} 2\pi(p + \nu_3)(1 - 2\sin^2 \pi(2\nu_1 - p - \nu_3))^{m_2} \,, \\ \mathcal{F}_3 &= [\cos \pi(2\nu_1 - p - \nu_3)\cos \pi(p - \nu_3) + \sin \pi(2\nu_1 - p - \nu_3)\sin \pi(p - \nu_3)]^{\alpha_4-\alpha_1} + \\ &\quad +[\cos \pi(2\nu_1 - p - \nu_3)\cos \pi(p - \nu_3) - \sin \pi(2\nu_1 - p - \nu_3)\sin \pi(p - \nu_3)]^{\alpha_4-\alpha_1} \,, \\ \mathcal{F}_4 &= [\cos \pi(2\nu_1 - p - \nu_3)\cos \pi(p + \nu_3) + \sin \pi(2\nu_1 - p - \nu_3)\sin \pi(p + \nu_3)]^{\alpha_3-\alpha_2} + \\ &\quad +[\cos \pi(2\nu_1 - p - \nu_3)\cos \pi(p + \nu_3) - \sin \pi(2\nu_1 - p - \nu_3)\sin \pi(p + \nu_3)]^{\alpha_3-\alpha_2} \,. \end{aligned} \tag{B.8}$$

We consider first the case in which $(\alpha_4 - \alpha_1)$ is odd and, consequently, $(\alpha_3 - \alpha_2)$ is even. $\mathcal{F}_3$ takes the form

$$\begin{aligned}\mathcal{F}_3 &= [I_1 + J_1]^{\alpha_4-\alpha_1} + [I_1 - J_1]^{\alpha_4-\alpha_1} \\ &= \sum_{n_4+m_4=\alpha_4-\alpha_1} b'_{n_4 m_4}[I_1^{n_4} J_1^{m_4} + I_1^{n_4}(-J_1)^{m_4}] \\ &= \sum_{l_4,h_4;\, 2l_4+1+2h_4=\alpha_4-\alpha_1} b_{l_4 h_4}[I_1^{2l_4+1} J_1^{2h_4}] \,. \end{aligned} \tag{B.9}$$

In the sum only the even powers of J_1 survive, and consequently the odd ones of I_1 ($(\alpha_4-\alpha_1)$ is odd). $\mathcal{F}_4$ is analogous, with the difference that $(\alpha_3 - \alpha_2)$ is even

$$\begin{aligned}\mathcal{F}_4 &= [I_2 + J_2]^{\alpha_3-\alpha_2} + [I_2 - J_2]^{\alpha_3-\alpha_2} \\ &= \sum_{l_3,h_3;\, 2l_3+2h_3=\alpha_3-\alpha_2} b_{l_3 h_3}[I_2^{2l_3} J_2^{2h_3}] \,. \end{aligned} \tag{B.10}$$

As a consequence $\mathcal{M}_1$ becomes

$$\begin{aligned}\mathcal{M}_1 = & \sum_{n_1m_1,n_2m_2,l_3h_3,l_4h_4} a_{n_1m_1}a_{n_2m_2}b_{l_3h_3}b_{l_4h_4} \cdot \\ & \cdot [\cos 2\pi(p-\nu_3)]^{n_1} \cdot [\cos 2\pi(p+\nu_3)]^{n_2} \cdot [\cos \pi(p-\nu_3)]^{2l_4+1} \cdot \\ & \cdot [\cos \pi(p+\nu_3)]^{2l_3} \cdot [\sin \pi(p-\nu_3)]^{2h_4} \cdot [\sin \pi(p+\nu_3)]^{2h_3} \cdot \\ & \cdot [1-2\sin^2 \pi(2\nu_1-p-\nu_3)]^{m_1+m_2} \cdot [1-\sin^2 \pi(2\nu_1-p-\nu_3)]^{l_3+l_4} \cdot \\ & \cdot [\sin \pi(2\nu_1-p-\nu_3)]^{2(h_3+h_4)} \cdot \cos \pi(2\nu_1-p-\nu_3)\,. \end{aligned} \tag{B.11}$$

The integral in $\int_p^1 d\nu_1$ of (B.11) is the sum of several integrals of the type:

$$\begin{aligned} g(\nu_3,p) \int_p^1 & [\sin \pi(2\nu_1-p-\nu_3)]^{2\gamma} \cos \pi(2\nu_1-p-\nu_3) d\nu_1 = \\ & = -\frac{g(\nu_3,p)}{2\pi(2\gamma+1)} [\sin^{2\gamma+1} \pi(p+\nu_3) + \sin^{2\gamma+1} \pi(p-\nu_3)]\,, \end{aligned} \tag{B.12}$$

with non-negative γ.

In turn the integral over ν_1 of (B.11) can be expressed as the sum of terms like the following ones with different coefficients:

$$\begin{aligned} \mathcal{T}_1 = & [\cos 2\pi(p-\nu_3)]^{n_1} \cdot [\cos 2\pi(p+\nu_3)]^{n_2} \cdot [\cos \pi(p-\nu_3)]^{2l_4+1} \cdot \\ & \cdot [\cos \pi(p+\nu_3)]^{2l_3} \cdot [\sin \pi(p-\nu_3)]^{2h_4} \cdot [\sin \pi(p+\nu_3)]^{2h_3} \cdot \\ & \cdot [\sin^{2\gamma+1} \pi(p+\nu_3) + \sin^{2\gamma+1} \pi(p-\nu_3)] \\ = & [\cos 2\pi p \cos 2\pi\nu_3 + \sin 2\pi p \sin 2\pi\nu_3]^{n_1} \cdot [\cos 2\pi p \cos 2\pi\nu_3 - \sin 2\pi p \sin 2\pi\nu_3]^{n_2} \cdot \\ & \cdot [\cos \pi p \cos \pi\nu_3 + \sin \pi p \sin \pi\nu_3]^{2l_4+1} \cdot [\cos \pi p \cos \pi\nu_3 - \sin \pi p \sin \pi\nu_3]^{2l_3} \cdot \\ & \cdot [\sin \pi p \cos \pi\nu_3 - \cos \pi p \sin \pi\nu_3]^{2h_4} \cdot [\sin \pi p \cos \pi\nu_3 + \cos \pi p \sin \pi\nu_3]^{2h_3} \cdot \\ & \cdot \{[\sin \pi p \cos \pi\nu_3 + \cos \pi p \sin \pi\nu_3]^{2\gamma+1} + [\sin \pi p \cos \pi\nu_3 - \cos \pi p \sin \pi\nu_3]^{2\gamma+1}\} \cdot \\ \equiv & [A+B]^{n_1} \cdot [A-B]^{n_2} \cdot [C+D]^{2l_4+1} \cdot [C-D]^{2l_3} \cdot \\ & \cdot [E-F]^{2h_4} \cdot [E+F]^{2h_3} \cdot \{G\}\,. \end{aligned} \tag{B.13}$$

We remember that $\mathcal{M}_1$ is multiplied by $[f_a(u) \cdot f_b(t)]$. To each of the terms like $\mathcal{T}_1$ an analogous one corresponds, belonging to the integral in ν_1 of $\mathcal{M}_2$, with the same coefficient, given by

$$\begin{aligned} \mathcal{T}_2 = & [A+B]^{n_2} \cdot [A-B]^{n_1} \cdot [C+D]^{2l_3} \cdot [C-D]^{2l_4+1} \cdot \\ & \cdot [E-F]^{2h_3} \cdot [E+F]^{2h_4} \cdot \{G\}\,, \end{aligned} \tag{B.14}$$

and multiplied by $[f_a(t) \cdot f_b(u)]$.

Now we perform the integral

$$\begin{aligned} \int_0^1 d\nu_3 e^{\frac{i}{2}k_3\theta k_4} e^{-ik_3\theta k_4 \nu_3} \int_0^1 dp e^{-\frac{i}{2}k_1\theta k_2} e^{ik_1\theta k_2\, p} \times \\ \times A_h(\sin \pi p, \sin \pi\nu_3; s,t,u)[f_a(u) \cdot f_b(t)\mathcal{T}_1 + f_a(t) \cdot f_b(u)\mathcal{T}_2]\,. \end{aligned} \tag{B.15}$$

These integrals are of the type

$$\begin{aligned}\int_0^1 dx e^{-\frac{i}{2}a} e^{iax} (\sin \pi x)^b &= \frac{2^{-b}\Gamma(1+b)}{\Gamma(1+\frac{b}{2}-\frac{a}{2\pi})\Gamma(1+\frac{b}{2}+\frac{a}{2\pi})} \\ \int_0^1 dx e^{-\frac{i}{2}a} e^{iax} (\sin \pi x)^b \cos \pi x &= \frac{-ia2^{-b-1}\Gamma(1+b)}{\pi\Gamma(\frac{3}{2}+\frac{b}{2}-\frac{a}{2\pi})\Gamma(\frac{3}{2}+\frac{b}{2}+\frac{a}{2\pi})}\,. \end{aligned} \tag{B.16}$$

We concentrate our attention on the integrals in p, because they are even functions of $k_1\theta k_2$ when the integrand contains even powers of $\cos \pi p$ and are odd otherwise. The integral in ν_3 does involve k_1 and k_2 only in their symmetric combination s.

We note that the factor G appears in both (B.13) and (B.14), and that it is a function of even powers of $\cos \pi p$; we could prove it by repeating the arguments we used in (B.9). Therefore we focus on the quantity

$$\begin{aligned}\mathcal{Q} = &[A+B]^{n_1}[A-B]^{n_2}[C+D]^{2l_4+1}[C-D]^{2l_3}[E-F]^{2h_4} \cdot \\ &\cdot [E+F]^{2h_3} \cdot [f_a(u)\cdot f_b(t)] + \\ &+ [A+B]^{n_2}[A-B]^{n_1}[C+D]^{2l_3}[C-D]^{2l_4+1}[E-F]^{2h_3} \cdot \\ &\cdot [E+F]^{2h_4} \cdot [f_a(t)\cdot f_b(u)]\,. \end{aligned} \tag{B.17}$$

This expression can be rewritten as

$$\begin{aligned}\mathcal{Q} = \sum_{N_i M_i} a_{N_i M_i} \Big\{ &A^{N_\mu+N_\nu} B^{M_\mu} (-B)^{M_\nu} C^{N_\omega+N_\lambda} D^{M_\omega} (-D)^{M_\lambda} \times \\ &\times E^{N_\rho+N_\sigma} (-F)^{M_\rho} F^{M_\sigma} \cdot [f_a(u)\cdot f_b(t)] + \\ &+ A^{N_\mu+N_\nu} (-B)^{M_\mu} B^{M_\nu} C^{N_\omega+N_\lambda} (-D)^{M_\omega} D^{M_\lambda} \times \\ &\times E^{N_\rho+N_\sigma} F^{M_\rho} (-F)^{M_\sigma} \cdot [f_a(t)\cdot f_b(u)] \Big\}\,, \end{aligned} \tag{B.18}$$

where i is a compact way to indicate $\mu, \nu, \omega, \lambda, \rho, \sigma$, and where the sum is over

$$\begin{aligned} N_\mu + M_\mu &= n_1\,, & N_\nu + M_\nu &= n_2\,, & N_\omega + M_\omega &= 2k_4+1\,, \\ N_\lambda + M_\lambda &= 2k_3\,, & N_\rho + M_\rho &= 2h_4\,, & N_\sigma + M_\sigma &= 2h_3\,. \end{aligned} \tag{B.19}$$

We notice that B, C and F contain the factor $\cos \pi p$. Taking the relation

$$A^a \cdot B^b \cdot C^c \cdot D^d \cdot E^e \cdot F^f = c(\sin \pi p, \cos \pi \nu_3, \sin \pi \nu_3) \cdot [\cos \pi p]^{b+c+f}$$

into account, eq. (B.18) can be rewritten as

$$\begin{aligned}\mathcal{Q} = \sum_{N_i M_i} &a_{N_i M_i} c_{N_i M_i} (\sin \pi p, \cos \pi \nu_3, \sin \pi \nu_3) \cdot [\cos \pi p]^{M\mu+M\nu+N\omega+N\lambda+M\rho+M\sigma} \cdot \\ &\cdot \Big[(-1)^{M\nu+M\lambda+M\rho} f_a(u) f_b(t) + (-1)^{M\mu+M\omega+M\sigma} f_a(t) f_b(u)\Big]\,. \end{aligned} \tag{B.20}$$

Now we can show why the quantity (B.4) vanishes after the symmetrization with respect to $k_1 \leftrightarrow k_2$.

- If $M_\nu + M_\lambda + M_\rho$ has the same parity of $M_\mu + M_\omega + M_\sigma$, then

1. the *symmetric* factor $[f_a(u)f_b(t) + f_a(t)f_b(u)]$ appears;
2. the exponent $M_\mu + M_\nu + N_\omega + N_\lambda + M_\rho + M_\sigma$ of $\cos\pi p$ is *odd*, because
   - if $M_\lambda + M_\omega$ is even, then
     (a) $N_\lambda + N_\omega$ is odd,
     (b) $(M_\nu + M_\rho) + (M_\mu + M_\sigma)$ is even;
   - if $M_\lambda + M_\omega$ is odd, then
     (a) $N_\lambda + N_\omega$ is even,
     (b) $(M_\nu + M_\rho) + (M_\mu + M_\sigma)$ is odd.

In this case we have that the integral in p is odd under the exchange $k_1 \leftrightarrow k_2$ and that it is multiplied by a factor even under such an exchange.

- If $M_\nu + M_\lambda + M_\rho$ has the opposite parity of $M_\mu + M_\omega + M_\sigma$, then

1. the *antisymmetric* factor $[f_a(u)f_b(t) - f_a(t)f_b(u)]$ appears;
2. the exponent $M_\mu + M_\nu + N_\omega + N_\lambda + M_\rho + M_\sigma$ of $\cos\pi p$ is *even*, because
   - if $M_\lambda + M_\omega$ is even, then
     (a) $N_\lambda + N_\omega$ is odd,
     (b) $(M_\nu + M_\rho) + (M_\mu + M_\sigma)$ is odd;
   - if $M_\lambda + M\omega$ is odd, then
     (a) $N_\lambda + N_\omega$ is even,
     (b) $(M_\nu + M_\rho) + (M_\mu + M_\sigma)$ is even.

In this case we have that the integral in p is even under the exchange $k_1 \leftrightarrow k_2$ and that it is multiplied by a factor which is odd.

The case in which $\alpha_4 - \alpha_1$ is even (and consequently $\alpha_3 - a_2$ is odd) can be treated in the same way.

C. The open-string threshold

The amplitude (3.14) is the sum of three graphs, according to the integration regions of the variables ν_i's. In the following we explicitly discuss the configuration reported in figure 1, namely eq. (3.14) with the integrations restricted to the region $1 \geq \nu_1 \geq \nu_2 \geq \nu_3 \geq 0$.

When $l \sim \infty$:

$$\eta\left(\frac{2\pi i\alpha'}{l}\right) \sim \left(\frac{2\pi\alpha'}{l}\right)^{-1/2} e^{-l/24\alpha'} ; \tag{C.1}$$

$$\vartheta_4\left(\nu, \frac{2\pi i\alpha'}{l}\right) \sim \left(\frac{2\pi\alpha'}{l}\right)^{-1/2} e^{\frac{l}{2\alpha'}(\nu-\nu^2)} e^{-\frac{l}{8\alpha'}} ; \tag{C.2}$$

$$\vartheta_1\left(\nu, \frac{2\pi i\alpha'}{l}\right) \sim \left(\frac{2\pi\alpha'}{l}\right)^{-1/2} e^{\frac{l}{2\alpha'}(\nu-\nu^2)} e^{-\frac{l}{8\alpha'}} . \tag{C.3}$$

Hence the asymptotic behaviour of (3.14) is

$$A_S^{as} = \mathcal{N}_\mathcal{S}\mathcal{K}\int^\infty dl\, l^{3-d/2} e^{-\frac{Y^2 l}{2\pi\alpha'}} e^{-\frac{i}{2}k_1\theta k_2} e^{\frac{i}{2}k_3\theta k_4} \times$$
$$\times \left(\prod_{n=1}^{3}\int_0^{\nu_{n-1}} d\nu_n\right) e^{i(\nu_{12})k_1\theta k_2} e^{-i\nu_3 k_3\theta k_4} e^{l\sum_{n<m=1}^4 \nu_{nm}(1-\nu_{nm})k_n G^{-1}k_m}\,. \tag{C.4}$$

We change the integration variables, introducing

$$\begin{aligned} x_1 &= \nu_1 - \nu_2 \\ x_2 &= \nu_2 - \nu_3 \\ x_3 &= \nu_3 \\ x_4 &= 1 - \nu_1\,, \end{aligned} \tag{C.5}$$

such that $x_1 + x_2 + x_3 + x_4 = 1$. In these new variables we get:

$$\begin{aligned} \sum_{n<m}^{4}\nu_{nm}(1-\nu_{nm})k_n G^{-1}k_m &= -2k_1G^{-1}k_2(x_2+x_3)x_4 - 2k_1G^{-1}k_3x_3x_4 - \\ &\quad - 2k_2G^{-1}k_3(x_1+x_4)x_3 - k_1G^{-1}k_1(x_1+x_2+x_3)x_4 - \\ &\quad - k_2G^{-1}k_2(x_1+x_3)(x_2+x_4) - k_3G^{-1}k_3(x_1+x_2+x_4)x_3\,. \end{aligned} \tag{C.6}$$

We impose the on-shell condition on the external momenta

$$k_n G^{-1} k_n = 0 \tag{C.7}$$

and introduce the Mandelstam invariants

$$\begin{aligned} s &= -(k_1+k_2)G^{-1}(k_1+k_2)\,, \\ t &= -(k_1+k_4)G^{-1}(k_1+k_4)\,, \\ u &= -(k_1+k_3)G^{-1}(k_1+k_3)\,. \end{aligned} \tag{C.8}$$

Considering these relations, eq. (C.6) becomes

$$\sum_{n<m}^{4}\nu_{nm}(1-\nu_{nm})k_n G^{-1}k_m = s(x_2+x_3)x_4 + ux_3x_4 + t(x_1+x_4)x_3 \tag{C.9}$$

and the amplitude (C.4) can be rewritten as

$$A_S^{as} = \mathcal{N}_\mathcal{S}\mathcal{K}\int^\infty dl\, l^{3-d/2} e^{-\frac{i}{2}k_1\theta k_2} e^{\frac{i}{2}k_3\theta k_4}\left(\prod_{n=1}^{4}\int_0^1 dx_n\right)\delta\left(1-\sum_n x_n\right)\times$$
$$\times\, e^{i(\nu_{12})k_1\theta k_2} e^{-i\nu_3k_3\theta k_4} e^{-l(m^2 - s(x_2+x_3)x_4 - ux_3x_4 - t(x_1+x_4)x_3)}\,, \tag{C.10}$$

where we have defined $m^2 \equiv Y^2/(2\pi\alpha')^2$. The function $A_S^{as}(s)$ is well defined when the integral converges, namely when

$$m^2 - s(x_2+x_3)x_4 - ux_3x_4 - t(x_1+x_4)x_3 > 0 \qquad \forall x_1, x_2, x_3, x_4\,. \tag{C.11}$$

This condition requires

$$s < \min_{\{x\}} \left[\frac{m^2}{(x_2+x_3)x_4} + (-u)\frac{x_3}{(x_2+x_3)} + (-t)\frac{(x_1+x_4)x_3}{(x_2+x_3)x_4} \right]$$
$$= m^2 \min_{0<x_2<1} \left[\frac{1}{x_2(1-x_2)} \right] = 4m^2\,. \tag{C.12}$$

We have thereby recovered the expected threshold condition. Analogous contribution is obtained by completing the integration region of the variable ν_3.

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